PHIL 1008 Problem Set 2

1. (20 marks)

True or false?
Circle 'T' if the statement is true.
Circle 'F' if the statement is false.
For this question, you should assume that φ is a WFF of MPL, and the derivations are in our natural deduction system for MPL.

- T (F) φ is true under at least one interpretation. eg. If φ is '$\forall x Px$'
- T (F) If c is a constant and v is a variable, then $\varphi v/c$ is a well-formed formula of MPL.
- (T) F Some correct derivations have more than 1 million lines.
- T (F) "$\exists x Ax$" is valid MPL formula.
- T (F) φ might be a well-formed formula of SL.
- T (F) If rule $\leftrightarrow I$ is removed, then the resulting system would not be sound.
- (T) F Some MPL sequents are not valid.
- (T) F If φ is valid, then there is a derivation of φ with no dependencies. b/c our system is complete.
- T (F) "$\forall x(\exists x Rx \& Cx)$" is a well-formed formula of MPL.
- T (F) The truth table method can always determine whether or not an MPL formula is valid.

/20

(2) (40 marks) Show each of the following using natural deduction:

(a) $\forall x(Ax \rightarrow Bx), \sim Ba \vdash \sim Aa$

1 ① ∀x(Ax → Bx) A
2 ② ~~Ba~~ A
3 ③ Aa A
1 ④ (Aa → Ba) 1 ∀E
1,3 ⑤ Ba 3,4 →E
1,2,3 ⑥ (Ba & ~Ba) 2,5 &I
1,2 ⑦ ~Aa 3,6 ~I

(b) $\forall x(Ax \leftrightarrow Bx), \forall x \sim (Bx \vee \sim Cx) \vdash \sim Ab$

1 1. ∀x (Ax ↔ Bx) A
2 2. ∀x ~(Bx v ~Cx) A
3 3. Ab A
1 4. (Ab ↔ Bb) 1 ∀E
1 5. (Ab → Bb) 4 ↔E
1,3 6. Bb 3,5 →E
1,3 7. (Bb v ~Cb) 6 vI
2 8. ~(Bb v ~Cb) 2 ∀E
1,2,3 9. ((Bb v ~Cb) & ~(Bb v ~Cb)) 7,8 &I
1,2 10. ~Ab 3 ~I

(c) $\sim\exists x \sim Ax \vdash \forall x Ax$

1 1. ~∃x ~Ax A
2 2. ~Aa A
2 3. ∃x ~Ax 2 ∃I
1,2 4. (~∃x ~Ax & ∃x ~Ax) 1,3 &I
1 5. Aa 2,4 ~E
1 6. ∀xAx 5 ∀I

(d) $\forall x(Ax \& Bx) \vdash (\forall x Ax \& \forall x Bx)$

1 1. ∀x (Ax & Bx) A
1 2. (Aa & Ba) 1 ∀E
1 3. Aa 2 &E
1 4. ∀xAx 3 ∀I
1 5. Ba 2 &E
1 6. ∀xBx 5 ∀I
1 7. (∀xAx & ∀x Bx) 4,6 &I

(e) $\forall x(Ax \rightarrow \exists yBy) \vdash (\exists xAx \rightarrow \exists yBy)$

1	(1)	$\forall x(Ax \rightarrow \exists yBy)$	A
2	(2)	$\exists xAx$	A
3	(3)	Aa	A
1	(4)	$(Aa \rightarrow \exists yBy)$	1 ∀E
1,3	(5)	$\exists yBy$	3,4 →E
1,2	(6)	$\exists yBy$	2,3,5 ∃E
1	(7)	$(\exists xAx \rightarrow \exists yBy)$	2,6 →I

(f) $\forall x(Ax \vee Gx) \vdash (\forall xAx \vee \exists xGx)$

1	(1)	$\forall x(Ax \vee Gx)$	A
2	(2)	$\neg(\forall xAx \vee \exists xGx)$	A
1	(3)	$(Aa \vee Ga)$	1 ∀E
4	(4)	$\neg Aa$	A
1,4	(5)	Ga	3,4 ∨E
1,4	(6)	$\exists xGx$	5 ∃I
1,4	(7)	$(\forall xAx \vee \exists xGx)$	6 ∨I
1,2,4	(8)	$((\forall xAx \vee \exists xGx) \& \neg(\forall xAx \vee \exists xGx))$	2,7 &I
1,2	(9)	Aa	4,8 ¬E
1,2	(10)	$\forall xAx$	9 ∀I
1,2	(11)	$(\forall xAx \vee \exists xGx)$	10 ∨I
1,2	(12)	$((\forall xAx \vee \exists xGx) \& \neg(\forall xAx \vee \exists xGx))$	2,11 &I
1	(13)	$(\forall xAx \vee \exists xGx)$	2,12 ¬E

(g) $\vdash \exists x(\exists yAy \rightarrow Ax)$

(h) $\forall x(Ax \vee (Bx \vee Cx)), \forall x(Bx \rightarrow Dx), \forall x(Cx \rightarrow Dx) \vdash \forall x(Ax \vee Dx)$

1	(1)	$\forall x((Ax \vee (Bx \vee Cx))$	A
2	(2)	$\forall x(Bx \rightarrow Dx)$	A
3	(3)	$\forall x(Cx \rightarrow Dx)$	A
1	(4)	$(Aa \vee (Ba \vee Ca))$	1 ∀E
5	(5)	$\neg Aa$	A
1,5	(6)	$(Ba \vee Ca)$	4,5 ∨E
2	(7)	$(Ba \rightarrow Da)$	2 ∀E
3	(8)	$(Ca \rightarrow Da)$	3 ∀E
1,2,3,5	(9)	$(Da \vee Da)$	6,7,8 PC
10	(10)	$\neg Da$	A
1,2,3,5,10	(11)	Da	9,10 ∨E
1,2,3,5,10	(12)	$(Da \& \neg Da)$	10,11 &I
1,2,3,5	(13)	Da	10,12 ¬E
14	(14)	Aa	A
	(15)	$(Aa \rightarrow Aa)$	14 →I
16	(16)	$\neg(Aa \vee \neg Aa)$	A
17	(17)	Aa	A
17	(18)	$(Aa \vee \neg Aa)$	17 ∨I
16,17	(19)	$((Aa \vee \neg Aa) \& \neg(Aa \vee \neg Aa))$	16,18 &I
16	(20)	$\neg Aa$	17,19 ¬I
16	(21)	$(Aa \vee \neg Aa)$	16 ∨I
16	(22)	$((Aa \vee \neg Aa) \& \neg(Aa \vee \neg Aa))$	16,21 &I
	(23)	$(Aa \vee \neg Aa)$	16,22 ¬E
1,2,3	(24)	$(\neg Aa \rightarrow Da)$	5,13 →I
1,2,3	(25)	$(Aa \vee Da)$	15,23,24 PC
1,2,3	(26)	$\forall x(Ax \vee Dx)$	25 ∀I

(i) $Ac, Bc, \forall x(Bx \rightarrow Cx) \vdash \exists x(Ax \& Cx)$

1	(1)	Ac	A
2	(2)	Bc	A
3	(3)	$\forall x(Bx \rightarrow Cx)$	A
3	(4)	$(Bc \rightarrow Cc)$	3 ∀E
2,3	(5)	Cc	2,4 →E
1,2,3	(6)	$(Ac \& Cc)$	1,5 &I
1,2,3	(7)	$\exists x(Ax \& Cx)$	6 ∃I

(j) $\forall x(Ax \rightarrow Bx) \vdash \sim\exists x(Ax \& \sim Bx)$

1	(1)	$\forall x(Ax \rightarrow Bx)$	A
2	(2)	$\exists x(Ax \& \sim Bx)$	A
3	(3)	$(Aa \& \sim Ba)$	A
3	(4)	Aa	3 &E
1	(5)	$(Aa \rightarrow Ba)$	1 ∀E
1,3	(6)	Ba	4,5 →E
3	(7)	$\sim Ba$	3 &E
8	(8)	$\sim(Cb \& \sim Cb)$	A
1,3	(9)	$(Ba \& \sim Ba)$	6,7 &I
1,3	(10)	$(Cb \& \sim Cb)$	8,9 ~E
1,2	(11)	$(Cb \& \sim Cb)$	2,3,10 ∃E
1	(12)	$\sim\exists x(Ax \& \sim Bx)$	2,11 ~I

/40

(3) (6 marks) Circle your answer.

Suppose the MPL natural deduction system is revised by adding the following rule: (NR) for any variable v and constant c, if you have derived $\forall v(\varphi v/c \vee \psi v/c)$, then you can write down φ, depending on everything $\forall v(\varphi v/c \vee \psi v/c)$ depends on.

Is the revised system sound? **YES** **NO**

Is the revised system complete? **YES** **NO**

/6

(4) (6 marks) Circle your answer.

Suppose rule PC is removed from the MPL natural deduction system.

Is the revised system sound? **YES** **NO**

Is the revised system complete? **YES** **NO**

/6

(5) (6 marks) Circle your answer.

Suppose the MPL natural deduction system is revised by adding the following rule: (NR2) for any variable v and constant c, if you have derived $\forall v \varphi v/c$, then you can write down $\exists v \varphi v/c$, depending on everything $\forall v \varphi v/c$ depends on.

Is the revised system sound?	**YES** (circled)	**NO**
Is the revised system complete?	**YES** (circled)	**NO**

/6

(6) (6 marks) Circle your answer.

Suppose rule $\forall E$ is removed from the MPL natural deduction system.

Is the revised system sound?	**YES** (circled)	**NO**
Is the revised system complete?	**YES**	**NO** (circled)

/6

(7) (16 marks)

Translate the following two arguments into MPL. (The arguments come from Lewis Carroll's Symbolic Logic.*) Be sure to write down your translation schemes. Then, for each argument, either show that the argument is valid using natural deduction, or show that the argument is not valid by giving an appropriate interpretation.*

1. Babies are illogical. Nobody is despised who can manage a crocodile. Illogical people are despised. Babies are people. Therefore. babies cannot manage crocodiles.

2. No terriers wander among the signs of the zodiac. Something that does not wander among the signs of the zodiac, is not a comet. Nothing but a terrier has a curly tail. Therefore some comet has a curly tail.

Valid

1) Px : x is a person
Bx : x is a baby
Ix : x is illogical
Dx : x is despised
Cx : x can manage a crocodile

1	(1)	$\forall x(Bx \to Ix)$	A
2	(2)	$\neg\exists x(Cx \& Dx)$	A
3	(3)	$\forall x((Ix \& Px) \to Dx)$	A
4	(4)	$\forall x(Bx \to Px)$	A
1	(5)	$(Ba \to Ia)$	1 $\forall$E
3	(6)	$((Ia \& Pa) \to Da)$	3 $\forall$E
4	(7)	$(Ba \to Pa)$	4 $\forall$E
8	(8)	Ba	A
1,8	(9)	Ia	5,8 $\to$E
4,8	(10)	Pa	7,8 $\to$E
1,4,8	(11)	$(Ia \& Pa)$	9,10 &I
1,3,4,8	(12)	Da	6,11 $\to$E
13	(13)	Ca	A
3,4,8,13	(14)	$(Ca \& Da)$	12,13 &I
1,3,4,8,13	(15)	$\exists x(Cx \& Dx)$	2,14 &I
	(16)	$(\exists x(Cx \& Dx) \& \neg\exists x(Cx \& Dx))$	2,15 &I
1,2,3,4,8	(17)	$\neg Ca$	13,16 $\neg$I
1,2,3,4	(18)	$(Ba \to \neg Ca)$	8,17 $\to$I
1,2,3,4	(19)	$\forall x(Bx \to \neg Cx)$	18 $\forall$I

2) $\neg\exists x(Ax \& Bx)$ Invalid.
$\forall x(\neg Bx \to \neg Cx)$
$\forall x(Dx \to Ax) \vdash \exists x(Cx \& Dx)$

Ax : x is a mammal
Bx : x is a fish
Cx : x has gill
Dx : x is a dog.

/16

2g)

1	1) ~∃y(∃xAx→Ay)	A
2	2) ~∀y~(∃xAx_Ay)	A
3	3) (∃xAx→Aa)	A
3	4) ∃y(∃xAx_Ay)	3 ∃I
1, 3	5) (∃y(∃xAx_Ay)&~ ∃y(∃xAx_Ay))	1, 4 &I
1	6) ~(∃xAx→Aa)	3, 5 ~I
1	7) ∀y~(∃xAx→Ay)	6 ∀I
1	8) ~(∃xAx_Aa)	7 ∀E
9	9) (~∃xAx v Aa)	A
10	10) ~Aa	A
9, 10	11) ~∃xAx	9, 10 vE
9	12) (~Aa_~∃xAx)	10, 11 _I
13	13) ∃xAx	A
14	14) ~Aa	A
9, 14	15) ~∃xAx	12, 14 _E
9, 13, 14	16) (∃xAx&~∃xAx)	13, 15 &I
9, 13	17) Aa	14, 16 ~E
9	18) (∃xAx_Aa)	13, 17 _I
1, 9	19) ((∃xAx_Aa)&~(∃xAx→Aa))	8, 18 &I
1	20) ~(~∃xAx v Aa)	9, 19 ~I
21	21) ~(∃xAx&~Aa)	A
22	22) ~∃xAx	A
22	23) (~∃xAx v Aa)	22 vI
1, 22	24) ((~∃xAx v Aa)&~(~∃xAx v Aa))	20, 23 &I
1	25) ∃xAx	22, 24 ~E
26	26) Aa	A
26	27) (~∃xAx v Aa)	26 vI
1, 26	28) ((~∃xAx v Aa)&~ (~∃xAx v Aa))	20, 27 &I
1	29) ~Aa	26, 28 ~I
1	30) (∃xAx&~Aa)	25, 29 &I
1, 21	31) ((∃xAx&~Aa)&~ (∃xAx&~Aa))	21, 30 &I
1	32) (∃xAx&~Aa)	21, 31 ~E
1	33) ∃xAx	32 &E
1	34) ~Aa	32 &E
35	35) Aa	A
36	36) ~(Ab&~Ab)	A
1, 35	37) (Aa&~Aa)	34, 35 &I
1, 35	38) (Ab&~Ab)	36, 37 ~E
1	39) (Ab&~Ab)	33, 35, 38
∃E		
	40) ∃y(∃xAx_Ay)	1, 39 ~E