

Natural Deduction in MPL

Examples

Example 1

$\forall xHx \vdash Ha$

1 1 $\forall xHx$ A
1 2 Ha 1, $\forall E$

Example 2

$\forall xHx, (Hc \rightarrow \exists xGx) \vdash \exists xGx$

1 1 $\forall xHx$ A
2 2 $(Hc \rightarrow \exists xGx)$ A
1 3 Hc 1, $\forall E$
1,2 4 $\exists xGx$ 2, 3 $\rightarrow E$

Defective attempt at $(\forall xFx \rightarrow Ga), Fa \vdash Ga$

1 1 $(\forall xFx \rightarrow Ga)$ A
2 2 Fa A
1 3 $(Fa \rightarrow Ga)$ 1, $\forall E$ (Mistake!!!)
1,2 4 Ga 2,3 $\rightarrow E$

Example 3

$\forall x(Hx \rightarrow Mx), \forall xHx \vdash \forall xMx$

1 1 $\forall x(Hx \rightarrow Mx)$ A
2 2 $\forall xHx$ A
1 3 $(Ha \rightarrow Ma)$ 1 $\forall E$
2 4 Ha 2 $\forall E$
1, 2 5 Ma 3, 4 $\rightarrow E$
1, 2 6 $\forall xMx$ 5 $\forall I$

Defective attempt at $(Ha \rightarrow Ma), Ha \vdash \forall xMx$

1 1 $(Ha \rightarrow Ma)$ A
2 2 Ha A
1, 2 3 Ma 1, 2 $\rightarrow E$
1, 2 4 $\forall xMx$ 3 $\forall I$ (Mistake!!!)

Example 4

$Ha \vdash \exists yHy$

1 1 Ha A
 1 2 $\exists yHy$ 1, $\exists I$

Defective attempt at $(Ab \rightarrow Dc) \vdash (Ab \rightarrow \exists xDx)$

1 1 $(Ab \rightarrow Dc)$ A
 1 2 $(Ab \rightarrow \exists xDx)$ 1, $\exists I$ (Mistake!!!)

Example 5

$(Ab \rightarrow Dc) \vdash (Ab \rightarrow \exists xDx)$

1 1 $(Ab \rightarrow Dc)$ A
 2 2 Ab A
 1,2 3 Dc 1,2 $\rightarrow E$
 1,2 4 $\exists xDx$ 3, $\exists I$
 1 5 $(Ab \rightarrow \exists xDx)$ 2,4 $\rightarrow I$

Example 6

$\exists x(Fx \wedge Gx) \vdash \exists xFx$

1 1 $\exists x(Fx \wedge Gx)$ A
 2 2 $(Fa \wedge Ga)$ A
 2 3 Fa 2, $\wedge E$
 2 4 $\exists xFx$ 3, $\exists I$
 1 1 $\exists xFx$ 1,2,4, $\exists E$

Defective attempt at $\exists xFx, Ga \vdash \exists x(Fx \wedge Gx)$

1 1 $\exists xFx$ A
 2 2 Ga A
 3 3 Fa A
 1 4 Fa 1, 3 $\exists E$ (Mistake!!!)
 1,2 5 $(Fa \wedge Ga)$ 4, 2 $\wedge I$
 1,2 6 $\exists x(Fx \wedge Gx)$ 5 $\exists I$

Defective attempt at $\exists xFx \vdash \forall xFx$

1 1 $\exists xFx$ A
 2 2 Fa A
 1 3 Fa 1, 2 $\exists E$ (Mistake!!!)
 1 4 $\forall xFx$ 3 $\forall I$

Defective attempt at $\exists xFx, Fa \vdash \forall xFx$

1 1 $\exists xFx$ A
 2 2 Fa A

2 4 $\forall xFx$ 3 $\forall I$ (Mistake!!!)

Example 7

$\forall x(Fx \rightarrow \forall yGy) \vdash \forall x\forall y(Fx \rightarrow Gy)$

1	1	$\forall x(Fx \rightarrow \forall yGy)$	A
1	2	$(Fa \rightarrow \forall yGy)$	1 $\forall E$
3	3	Fa	A
1, 3	4	$\forall yGy$	2, 3 $\rightarrow E$
1, 3	5	Gb	4 $\forall E$
1	6	$(Fa \rightarrow Gb)$	3, 5 $\rightarrow I$
1	7	$\forall y(Fa \rightarrow Gy)$	6 $\forall I$
1	8	$\forall x\forall y(Fx \rightarrow Gy)$	7 $\forall I$

Example 8

$\forall x(Px \rightarrow Qx), \forall x(Qx \rightarrow Px) \vdash \forall x(Px \leftrightarrow Qx)$

1	1	$\forall x(Px \rightarrow Qx)$	A
2	2	$\forall x(Qx \rightarrow Px)$	A
1	3	$(Pa \rightarrow Qa)$	1 $\forall E$
2	4	$(Qa \rightarrow Pa)$	2 $\forall E$
1, 2	5	$((Pa \rightarrow Qa) \wedge (Qa \rightarrow Pa))$	3, 4 $\wedge I$
1, 2	6	$(Pa \leftrightarrow Qa)$	5 $\leftrightarrow I$
1, 2	7	$\forall x(Px \leftrightarrow Qx)$	6 $\forall I$

Example 9

$(\exists xPx \rightarrow \forall x(Qx \rightarrow Rx)), (Pa \wedge Qa) \vdash Ra$

1	1	$(\exists xPx \rightarrow \forall x(Qx \rightarrow Rx))$	A
2	2	$(Pa \wedge Qa)$	A
2	3	Pa	2 $\wedge E$
2	4	Qa	2 $\wedge E$
2	5	$\exists xPx$	3 $\exists I$
1, 2	6	$\forall x(Qx \rightarrow Rx)$	1, 5 $\rightarrow E$
1, 2	7	$(Qa \rightarrow Ra)$	6 $\forall E$
1, 2	8	Ra	7, 4 $\rightarrow E$

Example 10

$(\forall x(Px \rightarrow Qx) \rightarrow \exists x(Rx \wedge Sx)), (\forall x(Px \rightarrow Sx) \wedge \forall x(Sx \rightarrow Qx)) \vdash \exists xSx$

1	1	$(\forall x(Px \rightarrow Qx) \rightarrow \exists x(Rx \wedge Sx))$	A
2	2	$(\forall x(Px \rightarrow Sx) \wedge \forall x(Sx \rightarrow Qx))$	A
2	3	$\forall x(Px \rightarrow Sx)$	2 $\wedge E$
2	4	$\forall x(Sx \rightarrow Qx)$	2 $\wedge E$

2	5 (Pa→Sa)	3 ∨E
2	6 (Sa→Qa)	4 ∨E
7	7 Pa	A
2, 7	8 Sa	5, 7 →E
2, 7	9 Qa	6, 8 →E
2	10 (Pa→Qa)	7, 9 →I
2	11 $\forall x(Px \rightarrow Qx)$	10 $\forall I$
1, 2	12 $\exists x(Rx \wedge Sx)$	1, 11 →E
13	13 (Rb∧Sb)	A
13	14 Sb	13 ∧E
13	15 $\exists xSx$	15 $\exists I$
1, 2	16 $\exists xSx$	12, 13, 15 $\exists E$

Example 11

$(\forall x(Px \wedge \neg Qx) \rightarrow \exists xRx), \neg \exists x(Qx \vee Rx) \vdash \neg \forall xPx$

1	1 $(\forall x(Px \wedge \neg Qx) \rightarrow \exists xRx)$	A
2	2 $\neg \exists x(Qx \vee Rx)$	A
3	3 $\forall xPx$	A
3	4 Pa	3 $\forall E$
5	5 Qa	A
5	6 (Qa∨Ra)	5 $\vee I$
5	7 $\exists x(Qx \vee Rx)$	6 $\exists I$
2, 5	8 $(\exists x(Qx \vee Rx) \wedge \neg \exists x(Qx \vee Rx))$	7, 2 $\wedge I$
2	9 $\neg Qa$	5, 8 $\neg I$
2, 3	10 (Pa∧¬Qa)	4, 9 $\wedge I$
2, 3	11 $\forall x(Px \wedge \neg Qx)$	10 $\forall I$
1, 2, 3	12 $\exists xRx$	1, 11 →E
13	13 Rb	A
13	14 (Qb∨Rb)	13 $\vee I$
13	15 $\exists x(Qx \vee Rx)$	14 $\exists I$
1, 2, 3	16 $\exists x(Qx \vee Rx)$	12, 13, 15 $\exists E$
1, 2, 3	17 $(\exists x(Qx \vee Rx) \wedge \neg \exists x(Qx \vee Rx))$	16, 2 $\wedge I$
1, 2	18 $\neg \forall xPx$	3, 17 $\neg I$

Example 12

$(\exists x \neg Px \rightarrow \forall x \neg Qx), (\exists x \neg Px \rightarrow \exists x Qx), \forall x(Px \rightarrow Rx) \vdash \forall xRx$

1	1 $(\exists x \neg Px \rightarrow \forall x \neg Qx)$	A
2	2 $(\exists x \neg Px \rightarrow \exists x Qx)$	A
3	3 $\forall x(Px \rightarrow Rx)$	A
4	4 $\neg Pa$	A
4	5 $\exists x \neg Px$	4 $\exists I$

1, 4	6 $\forall x \neg Qx$	1, 5 $\rightarrow E$
2, 4	7 $\exists x Qx$	2, 5 $\rightarrow E$
8	8 Qb	7 A
1, 4	9 $\neg Qb$	6 $\forall E$
10	10 $\neg(Qc \wedge \neg Qc)$	A
1, 4, 8	11 $(Qb \wedge \neg Qb)$	8, 9 $\wedge I$
1, 4, 8	12 $(Qc \wedge \neg Qc)$	10, 11 $\neg E$
1, 2, 4	13 $(Qc \wedge \neg Qc)$	7, 8, 12 $\exists E$
1, 2	14 Pa	4, 13 $\neg E$
3	15 $(Pa \rightarrow Ra)$	3 $\forall E$
1, 2, 3	16 Ra	14, 15 $\rightarrow E$
1, 2, 3	17 $\forall x Rx$	16 $\forall I$

Example 13

$\neg \exists x (Px \vee Qx), (\exists x Rx \rightarrow \exists x Px), (\exists x Sx \rightarrow \exists x Qx) \vdash \neg \exists x (Rx \vee Sx)$

1	1 $\neg \exists x (Px \vee Qx)$	A
2	2 $(\exists x Rx \rightarrow \exists x Px)$	A
3	3 $(\exists x Sx \rightarrow \exists x Qx)$	A
4	4 $\exists x (Rx \vee Sx)$	A
5	5 $(Ra \vee Sa)$	A
6	6 Ra	A
6	7 $\exists x Rx$	6 $\exists I$
2, 6	8 $\exists x Px$	2, 7 $\rightarrow E$
9	9 Pb	A
9	10 $(Pb \vee Qb)$	9 $\vee I$
9	11 $\exists x (Px \vee Qx)$	10 $\exists I$
2, 6	12 $\exists x (Px \vee Qx)$	8, 9, 11 $\exists E$
1, 2, 6	13 $(\exists x (Px \vee Qx) \wedge \neg \exists x (Px \vee Qx))$	12, 1 $\wedge I$
1, 2	14 $\neg Ra$	6, 13 $\neg I$
1, 2, 5	15 Sa	5, 14 $\vee E$
1, 2, 5	16 $\exists x Sx$	15 $\exists I$
1, 2, 4	17 $\exists x Sx$	4, 5, 16 $\exists E$
1, 2, 3, 4	18 $\exists x Qx$	3, 17 $\rightarrow E$
19	19 Qc	A
19	20 $(Pc \vee Qc)$	19 $\vee I$
19	21 $\exists x (Px \vee Qx)$	20 $\exists I$
1, 2, 3, 4	22 $\exists x (Px \vee Qx)$	18, 19, 21 $\exists E$
1, 2, 3, 4	23 $(\exists x (Px \vee Qx) \wedge \neg \exists x (Px \vee Qx))$	22, 1 $\wedge I$
1, 2, 3	24 $\neg \exists x (Rx \vee Sx)$	4, 23 $\neg I$