

Topic DS: Answers to Exercises

Exercise 1.4a

Show the following using only &E, \rightarrow E, or A.

$(A \& B) \vdash A$

1 1. $(A \& B)$ A
1 2. A 1 &E

$((A \vee B) \& (B \rightarrow A)) \vdash A$

Cannot be shown with only these three rules.

$(A \rightarrow (B \& C)), A \vdash B$

1 1. $(A \rightarrow (B \& C))$ A
2 2. A A
1, 2 3. $(B \& C)$ 1, 2 \rightarrow E
1, 2 4. B 3 &E

$A, B \vdash (A \& B)$

Cannot be shown with only these three rules.

$A, (A \rightarrow (B \rightarrow (C \rightarrow (D \rightarrow (E \rightarrow F)))) \vdash F$

Cannot be shown.

Exercise 2.1a

Yes, this is a correct derivation.

Exercise 2.1b

Show $(A \& B) \vdash (B \& A)$

1 1. $(A \& B)$ A
1 2. B 1 &E
1 3. A 1 &E
1 4. $(B \& A)$ 2,3 &I

Exercise 2.1c

There is no longest derivation in our system.
Any derivation can be lengthened, for example
by using Rule A.

Exercise 2.2aShow $(A \leftrightarrow B), (B \& C) \vdash A$

| | | |
|------|---|-----------------------|
| 1 | 1. $(A \leftrightarrow B)$ | A |
| 2 | 2. $(B \& C)$ | A |
| 2 | 3. B | 2 &E |
| 1 | 4. $((A \rightarrow B) \& (B \rightarrow A))$ | 1 \leftrightarrow E |
| 1 | 5. $(B \rightarrow A)$ | 4 &E |
| 1, 2 | 6. A | 3, 5 \rightarrow E |

Show $(A \rightarrow A) \vdash (A \leftrightarrow A)$

| | | |
|---|---|-----------------------|
| 1 | 1. $(A \rightarrow A)$ | A |
| 1 | 2. $((A \rightarrow A) \& (A \rightarrow A))$ | 1, 1 &I |
| 1 | 3. $(A \leftrightarrow A)$ | 2 \leftrightarrow I |

Show $(A \leftrightarrow B), (B \leftrightarrow C), A \vdash C$

| | | |
|---------|---|-----------------------|
| 1 | 1. $(A \leftrightarrow B)$ | A |
| 2 | 2. $(B \leftrightarrow C)$ | A |
| 3 | 3. A | A |
| 1 | 4. $((A \rightarrow B) \& (B \rightarrow A))$ | 1 \leftrightarrow E |
| 1 | 5. $(A \rightarrow B)$ | 4 &E |
| 1, 3 | 6. B | 3, 5 \rightarrow E |
| 2 | 7. $((B \rightarrow C) \& (C \rightarrow B))$ | 2 \leftrightarrow E |
| 2 | 8. $(B \rightarrow C)$ | 7 &E |
| 1, 2, 3 | 9. C | 6, 8 \rightarrow E |

Exercise 2.3a

There is no example.

Any formula which is assumed (written down in a derivation using Rule A) is derived (written down in a derivation).

Exercise 2.3b

Yes, this is a correct derivation.

Exercise 2.3c

This is not a correct derivation.

On line 5, Rule $\rightarrow I$ is misused because line 3 is not an assumed.

Exercise 2.3d

Show $(P \rightarrow (Q \rightarrow R)) \vdash ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

| | | |
|---------|--|----------------------|
| 1 | 1. $(P \rightarrow (Q \rightarrow R))$ | A |
| 2 | 2. $(P \rightarrow Q)$ | A |
| 3 | 3. P | A |
| 2, 3 | 4. Q | 2, 3 $\rightarrow E$ |
| 1, 3 | 5. $(Q \rightarrow R)$ | 1, 3 $\rightarrow E$ |
| 1, 2, 3 | 6. R | 4, 5 $\rightarrow E$ |
| 1, 2 | 7. $(P \rightarrow R)$ | 3, 6 $\rightarrow I$ |
| 1 | 8. $((P \rightarrow Q) \rightarrow (P \rightarrow R))$ | 2, 7 $\rightarrow I$ |

Show $\vdash (A \rightarrow A)$

| | | |
|---|------------------------|----------------------|
| 1 | 1. A | A |
| | 2. $(A \rightarrow A)$ | 1, 1 $\rightarrow I$ |

Show $\vdash ((A \& B) \rightarrow A)$

| | | |
|---|-------------------------------|----------------------|
| 1 | 1. $(A \& B)$ | A |
| 1 | 2. A | 1 $\&E$ |
| | 3. $((A \& B) \rightarrow A)$ | 1, 2 $\rightarrow I$ |

Exercise 3.1a

Show $(B \& \sim B) \vdash A$

| | | |
|---|--------------------|---------------|
| 1 | 1. $(B \& \sim B)$ | A |
| 1 | 2. B | 1 $\&E$ |
| 1 | 3. $\sim B$ | 1 $\&E$ |
| 1 | 4. $(B \vee A)$ | 2 $\vee I$ |
| 1 | 5. A | 3, 4 $\vee E$ |

Exercise 3.1b

Yes, for example:

| | | |
|------|------------------------|---------|
| 1 | 1. $(A \vee A)$ | A |
| 2 | 2. $(A \rightarrow B)$ | A |
| 1, 2 | 3. $(B \vee B)$ | 1, 2 PC |

Exercise 3.1c

Show $((P \& P) \vee (Q \& Q)) \vdash (P \vee Q)$

| | | |
|---|-------------------------------|----------------------|
| 1 | 1. $((P \& P) \vee (Q \& Q))$ | A |
| 2 | 2. $(P \& P)$ | A |
| 2 | 3. P | 2 $\&E$ |
| | 4. $((P \& P) \rightarrow P)$ | 2, 3 $\rightarrow I$ |

| | | |
|---|--------------|----------|
| 5 | 5. (Q&Q) | A |
| 5 | 6. Q | 5 &E |
| | 7. ((Q&Q)→Q) | 5,6 →I |
| 1 | 8. (P∨Q) | 1,4,7 PC |

Exercise 3.1d

Show $(P\vee P), (P\rightarrow Q) \vdash (P\vee Q)$

| | | |
|------|----------|------------|
| 1 | 1. (P∨P) | A |
| 2 | 2. (P→Q) | A |
| 3 | 3. P | A |
| | 4. (P→P) | 3, 3 →I |
| 1, 2 | 5. (P∨Q) | 1, 2, 4 PC |

Exercise 3.2a

Show $(\sim A\vee\sim B) \vdash \sim(A\&B)$

| | | |
|------|---------------------------|---------------|
| 1 | 1. ($\sim A\vee\sim B$) | A |
| 2 | 2. (A&B) | A |
| 2 | 3. A | 2 &E |
| 4 | 4. $\sim A$ | A |
| 2, 4 | 5. (A& $\sim A$) | 3, 4 &I |
| 2 | 6. $\sim\sim A$ | 4, 5 \sim I |
| 1, 2 | 7. $\sim B$ | 1, 6 \vee E |
| 2 | 8. B | 2 &E |
| 1, 2 | 9. (B& $\sim B$) | 7, 8 &I |
| 1 | 10. $\sim(A\&B)$ | 2, 9 \sim I |

Exercise 3.2b

Show $\sim\sim A \vdash A$

| | | |
|------|-----------------------------|---------------|
| 1 | 1. $\sim\sim A$ | A |
| 2 | 2. $\sim A$ | A |
| 1, 2 | 3. ($\sim A\&\sim\sim A$) | 1, 2 &I |
| 1 | 4. A | 2, 3 \sim E |

Show $\vdash (\sim\sim A\rightarrow A)$

| | | |
|------|----------------------------------|---------------|
| 1 | 1. $\sim\sim A$ | A |
| 2 | 2. $\sim A$ | A |
| 1, 2 | 3. ($\sim A\&\sim\sim A$) | 1, 2 &I |
| 1 | 4. A | 2, 3 \sim E |
| | 5. ($\sim\sim A\rightarrow A$) | 1, 4 →I |

Show $\sim(P\vee Q) \vdash (\sim P\&\sim Q)$

| | | |
|---|--------------------|------------|
| 1 | 1. $\sim(P\vee Q)$ | A |
| 2 | 2. P | A |
| 2 | 3. (P∨Q) | 2 \vee I |

| | | |
|------|-------------------------------------|---------------|
| 1, 2 | 4. $((P \vee Q) \& \sim(P \vee Q))$ | 1, 3 &I |
| 1 | 5. $\sim P$ | 2, 4 $\sim I$ |
| 6 | 6. Q | A |
| 6 | 7. $(P \vee Q)$ | 6 $\vee I$ |
| 1, 6 | 8. $((P \vee Q) \& \sim(P \vee Q))$ | 1, 7 &I |
| 1 | 9. $\sim Q$ | 6, 8 $\sim I$ |
| 1 | 10. $(\sim P \& \sim Q)$ | 5, 9 &I |

Exercise 4.2

Yes. Other possible rules are: $\rightarrow E$, $\& E$ and $\vee E$.

Exercise 4.4

$\sim((A \& B) \vee C) \vdash (\sim(A \& B) \& \sim C)$

Just take the derivation for $\sim(P \vee Q) \vdash (\sim P \& \sim Q)$, and substitute every occurrence of "P" with "(A&B)" and "Q" with "C".

| | | |
|------|---|---------------|
| 1 | 1. $\sim((A \& B) \vee C)$ | A |
| 2 | 2. $(A \& B)$ | A |
| 2 | 3. $((A \& B) \vee C)$ | 2 $\vee I$ |
| 1, 2 | 4. $((A \& B) \vee C) \& \sim((A \& B) \vee C)$ | 1, 3 &I |
| 1 | 5. $\sim(A \& B)$ | 2, 4 $\sim I$ |
| 6 | 6. C | A |
| 6 | 7. $((A \& B) \vee C)$ | 6 $\vee I$ |
| 1, 6 | 8. $((A \& B) \vee C) \& \sim((A \& B) \vee C)$ | 1, 7 &I |
| 1 | 9. $\sim C$ | 6, 8 $\sim I$ |
| 1 | 10. $(\sim(A \& B) \& \sim C)$ | 5, 9 &I |

Exercise 4.5a: see the following pages

Show the following:

$(P \rightarrow Q), (R \rightarrow Q) \vdash ((P \vee R) \rightarrow Q)$

| | | |
|---------|---------------------------------|---------------------|
| 1 | 1. $(P \rightarrow Q)$ | A |
| 2 | 2. $(R \rightarrow Q)$ | A |
| 3 | 3. $(P \vee R)$ | A |
| 1,2,3 | 4. $(Q \vee Q)$ | 1,2,3 PC |
| 5 | 5. $\sim Q$ | A |
| 1,2,3,5 | 6. Q | 4,5 $\vee E$ |
| 1,2,3,5 | 7. $(Q \& \sim Q)$ | 5,6 $\&I$ |
| 1,2,3 | 8. Q | 5,7 $\sim E$ |
| 1,2 | 9. $((P \vee R) \rightarrow Q)$ | 3,8 $\rightarrow I$ |

$(A \& B), (C \& D) \vdash (A \& D)$

| | | |
|-----|---------------|-----------|
| 1 | 1. $(A \& B)$ | A |
| 2 | 2. $(C \& D)$ | A |
| 1 | 3. A | 1 $\&E$ |
| 2 | 4. D | 2 $\&E$ |
| 1,2 | 5. $(A \& D)$ | 3,4 $\&I$ |

$(P \& (Q \& R)), S \vdash (R \& S)$

| | | |
|-----|----------------------|-----------|
| 1 | 1. $(P \& (Q \& R))$ | A |
| 2 | 2. S | A |
| 1 | 3. $(Q \& R)$ | 1 $\&E$ |
| 1 | 4. R | 3 $\&E$ |
| 1,2 | 5. $(R \& S)$ | 4,2 $\&I$ |

$(P \rightarrow Q), (Q \rightarrow R), P \vdash R$

| | | |
|-------|------------------------|---------------------|
| 1 | 1. $(P \rightarrow Q)$ | A |
| 2 | 2. $(Q \rightarrow R)$ | A |
| 3 | 3. P | A |
| 1,3 | 4. Q | 1,3 \rightarrow E |
| 1,2,3 | 5. R | 2,4 \rightarrow E |

$(P \rightarrow (Q \rightarrow R)), (P \& Q) \vdash R$

| | | |
|-----|--|---------------------|
| 1 | 1. $(P \rightarrow (Q \rightarrow R))$ | A |
| 2 | 2. $(P \& Q)$ | A |
| 2 | 3. P | 2 &E |
| 1,2 | 4. $(Q \rightarrow R)$ | 1,3 \rightarrow E |
| 2 | 5. Q | 2 &E |
| 1,2 | 6. R | 4,5 \rightarrow E |

$(P \rightarrow (Q \rightarrow R)), Q \vdash (P \rightarrow R)$

| | | |
|-------|--|---------------------|
| 1 | 1. $(P \rightarrow (Q \rightarrow R))$ | A |
| 2 | 2. Q | A |
| 3 | 3. P | A |
| 1,3 | 4. $(Q \rightarrow R)$ | 1,3 \rightarrow E |
| 1,2,3 | 5. R | 2,4 \rightarrow E |
| 1,2 | 6. $(P \rightarrow R)$ | 3,5 \rightarrow I |

$((P \& Q) \rightarrow R) \vdash (P \rightarrow (Q \rightarrow R))$

| | | |
|-------|-------------------------------|---------------------|
| 1 | 1. $((P \& Q) \rightarrow R)$ | A |
| 2 | 2. P | A |
| 3 | 3. Q | A |
| 2,3 | 4. $(P \& Q)$ | 2,3 &I |
| 1,2,3 | 5. R | 1,4 \rightarrow E |
| 1,2 | 6. $(Q \rightarrow R)$ | 3,5 \rightarrow I |

1 7. $(P \rightarrow (Q \rightarrow R))$ 2,6 \rightarrow I

$(P \rightarrow (Q \rightarrow R)) \vdash ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

| | | |
|-------|--|---------------------|
| 1 | 1. $(P \rightarrow (Q \rightarrow R))$ | A |
| 2 | 2. $(P \rightarrow Q)$ | A |
| 3 | 3. P | A |
| 2,3 | 4. Q | 2,3 \rightarrow E |
| 1,3 | 5. $(Q \rightarrow R)$ | 1,3 \rightarrow E |
| 1,2,3 | 6. R | 4,5 \rightarrow E |
| 1,2 | 7. $(P \rightarrow R)$ | 3,6 \rightarrow I |
| 1 | 8. $((P \rightarrow Q) \rightarrow (P \rightarrow R))$ | 2,7 \rightarrow I |

$P \vdash (Q \rightarrow P)$

| | | |
|---|------------------------|---------------------|
| 1 | 1. P | A |
| 2 | 2. Q | A |
| 1 | 3. $(Q \rightarrow P)$ | 2,1 \rightarrow I |

$(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R)$

| | | |
|-------|------------------------|---------------------|
| 1 | 1. $(P \rightarrow Q)$ | A |
| 2 | 2. $(Q \rightarrow R)$ | A |
| 3 | 3. P | A |
| 1,3 | 4. Q | 1,3 \rightarrow E |
| 1,2,3 | 5. R | 2,4 \rightarrow E |
| 1,2 | 6. $(P \rightarrow R)$ | 3,5 \rightarrow I |

$\sim(P \& \sim Q) \vdash (P \rightarrow Q)$

| | | |
|-----|------------------------|------------|
| 1 | 1. $\sim(P \& \sim Q)$ | A |
| 2 | 2. P | A |
| 3 | 3. $\sim Q$ | A |
| 2,3 | 4. $(P \& \sim Q)$ | 2,3 $\&$ I |

| | | |
|-------|---|---------------------|
| 1,2,3 | 5. $((P \ \& \ \sim Q) \ \& \ \sim(P \ \& \ \sim Q))$ | 1,4 &I |
| 1,2 | 6. Q | 3,5 ~E |
| 1 | 7. $(P \rightarrow Q)$ | 2,6 \rightarrow I |

$P, \sim P \vdash \sim Q$

| | | |
|-----|------------------------|--------|
| 1 | 1. P | A |
| 2 | 2. $\sim P$ | A |
| 3 | 3. Q | A |
| 1,2 | 4. $(P \ \& \ \sim P)$ | 1,2 &I |
| 1,2 | 5. $\sim Q$ | 3,4 ~I |

$(\sim P \rightarrow \sim Q) \vdash (Q \rightarrow P)$

| | | |
|-------|----------------------------------|---------------------|
| 1 | 1. $(\sim P \rightarrow \sim Q)$ | A |
| 2 | 2. Q | A |
| 3 | 3. $\sim P$ | A |
| 1,3 | 4. $\sim Q$ | 1,3 \rightarrow E |
| 1,2,3 | 5. $(Q \ \& \ \sim Q)$ | 2,4 &I |
| 1,2 | 6. P | 3,5 ~E |
| 1 | 7. $(Q \rightarrow P)$ | 2,6 \rightarrow I |

$((P \vee Q) \rightarrow R) \vdash (P \rightarrow R)$

| | | |
|-----|---------------------------------|---------------------|
| 1 | 1. $((P \vee Q) \rightarrow R)$ | A |
| 2 | 2. P | A |
| 2 | 3. $(P \vee Q)$ | 2 \vee I |
| 1,2 | 4. R | 1,3 \rightarrow E |
| 1 | 5. $(P \rightarrow R)$ | 2,4 \rightarrow I |

$\sim(P \rightarrow Q) \vdash \sim(\sim P \vee Q)$

| | | |
|---|----------------------------|---|
| 1 | 1. $\sim(P \rightarrow Q)$ | A |
| 2 | 2. $(\sim P \vee Q)$ | A |

| | | |
|-----|---|---------------------|
| 3 | 3. P | A |
| 4 | 4. $\sim P$ | A |
| 3,4 | 5. $(P \& \sim P)$ | 3,4 &I |
| 3 | 6. $\sim\sim P$ | 4,5 \sim I |
| 2,3 | 7. Q | 2,6 \vee E |
| 2 | 8. $(P \rightarrow Q)$ | 3,7 \rightarrow I |
| 1,2 | 9. $((P \rightarrow Q) \& \sim(P \rightarrow Q))$ | 1,8 &I |
| 1 | 10. $\sim(\sim P \vee Q)$ | 2,9 \sim I |

$(P \& (Q \vee R)) \vdash ((P \& Q) \vee (P \& R))$

| | | |
|-----|--------------------------------|---------------------|
| 1 | 1. $(P \& (Q \vee R))$ | A |
| 1 | 2. P | 1 &E |
| 1 | 3. $(Q \vee R)$ | 1 &E |
| 4 | 4. Q | A |
| 1,4 | 5. $(P \& Q)$ | 2,4 &I |
| 6 | 6. R | A |
| 1,6 | 7. $(P \& R)$ | 2,6 &I |
| 1 | 8. $(Q \rightarrow (P \& Q))$ | 4,5 \rightarrow I |
| 1 | 9. $(R \rightarrow (P \& R))$ | 6,7 \rightarrow I |
| 1 | 10. $((P \& Q) \vee (P \& R))$ | 3,8,9 PC |

$(P \leftrightarrow \sim Q) \vdash \sim(P \leftrightarrow Q)$

| | | |
|-------|---|-----------------------|
| 1 | 1. $(P \leftrightarrow \sim Q)$ | A |
| 2 | 2. $(P \leftrightarrow Q)$ | A |
| 2 | 3. $((P \rightarrow Q) \& (Q \rightarrow P))$ | 2 \leftrightarrow E |
| 2 | 4. $(P \rightarrow Q)$ | 3 &E |
| 1 | 5. $((P \rightarrow \sim Q) \& (\sim Q \rightarrow P))$ | 1 \leftrightarrow E |
| 1 | 6. $(P \rightarrow \sim Q)$ | 5 &E |
| 7 | 7. P | A |
| 2,7 | 8. Q | 4,7 \rightarrow E |
| 1,7 | 9. $\sim Q$ | 6,7 \rightarrow E |
| 1,2,7 | 10. $(Q \& \sim Q)$ | 8,9 &I |
| 1,2 | 11. $\sim P$ | 7,10 \sim I |

| | | |
|--------|--|-------------------|
| 12 | 12. $\sim(Q \vee \sim Q)$ | A |
| 13 | 13. Q | A |
| 13 | 14. $(Q \vee \sim Q)$ | 13 $\vee I$ |
| 12,13 | 15. $((Q \vee \sim Q) \& \sim(Q \vee \sim Q))$ | 12,14 $\&I$ |
| 12 | 16. $\sim Q$ | 13,15 $\sim I$ |
| 12 | 17. $(Q \vee \sim Q)$ | 16 $\vee I$ |
| 12 | 18. $((Q \vee \sim Q) \& \sim(Q \vee \sim Q))$ | 12,17 $\&I$ |
| | 19. $(Q \vee \sim Q)$ | 12,18 $\sim E$ |
| 2 | 20. $(Q \rightarrow P)$ | 3 $\rightarrow E$ |
| 1 | 21. $(\sim Q \rightarrow P)$ | 5 $\rightarrow E$ |
| 1,2 | 22. $(P \vee P)$ | 19,20,21 PC |
| 23 | 23. $\sim P$ | A |
| 1,2,23 | 24. P | 22,23 $\vee E$ |
| 1,2,23 | 25. $(P \& \sim P)$ | 23,24 $\&I$ |
| 1,2 | 26. P | 23,25 $\sim E$ |
| 1,2 | 27. $(P \& \sim P)$ | 11,26 $\&I$ |
| 1 | 28. $\sim(P \leftrightarrow Q)$ | 2,27 $\sim I$ |