# Basic Modal Logic

Seminar 6

PHIL2520 Philosophy of Logic

13 November 2012

# Administration

- Assignment 2: Due Monday 26 November (hand into the philosophy office by 5 PM)
- Note: **No late assignments will be accepted**
- Pick up assignments from philosophy office from wed 28 Nov
- Questions for the essay will now be handed out next week (rather than this week)

Required reading for this seminar: Ch2, Priest

# The language M

Let M be the language obtained from S* by adding ‘□’ and ‘◊’, where

i) ‘□’ symbolises ‘necessarily’,

ii) ‘◊’ symbolises ‘possibly’, and

iii) ‘□A’ and ‘◊A’ are sentences in M if A is a sentence in M.

Solution to the above deficiency: Study logical consequence for M

# The possible worlds analysis of modality

- A is true at a world w iff A would have been true if A obtained
- ‘□A’ is true at w iff, for **any** possible world w’ that is possible relative to w, A is true at w’
- ‘◊A’ is true at w iff, for **some** possible world w’ that is possible relative to w, A is true at w’

[Whiteboard example 1]

# Logical consequence on M

Original Def: Q is a logical consequence of P1,…,Pn iff, for any meaning interpretation m, **necessarily**, if P1,…, Pn are true under m, then Q is true under m

Def (in terms of possible worlds): Q is a logical consequence of P1,…,Pn iff, for any meaning interpretation m, **for any possible world w**, if P1,…, Pn are true **at w** under m, then Q is true **at w** under m

# Binary set-relations

Def: A binary set-relation on a set Z is a set of ordered pairs, each pair consisting of two members of Z (e.g., {<x1,y1>, <x2,y2>, <x3,y3>}

Def: x stands in a binary set-relation R to y iff <x,y> ∈ R

Example: The set-relation of **being smaller than** on the set {0,1,2} is {<0,1>, <1,2>, <2,3>}. An x stands in this relation to a y iff <x,y> ∈ {<0,1>, <1,2>, <0,2>}.

# K models

A K model of M is a triple <W, R,v>, where

i) W is a non-empty set (the set of “possible worlds”),

ii) R is a binary set-relation on W (the “relative possibility relation”), and

iii) v is a function mapping each pair comprising of a sentential constant p and world w to either 0 or 1 (denoted by $v_w(p)$).

Note: ‘K’ stands for Saul Kripke

# K models (cont)

v is called the interpretation wrt <W, R, v>.

Given an interpretation v, v is extended to apply to all sentences in M so that it maps all sentences to either 1 or 0 in such a way that:

i) $v_w(\sim A)=1$ iff $v_w(A)=0$

ii) $v_w(A\&B)=1$ iff $v_w(A)=1$ and $v_w(B)=1$

iii) $v_w(A \vee B)=1$ iff either $v_w(A)=1$ or $v_w(B)=1$

iv) $v_w(A \supset B)=1$ iff either $v_w(A)=0$ or $v_w(B)=1$

v) $v_w(A \equiv B)=1$ iff $v_w(A)=v_w(B)$

# K models (cont)

**vi) $v_w(\Box A)=1$ iff, for any w’ in W such that w stands in R to w’, $v_{w'}(A)=1$, and**

**vii) $v_w(\Diamond A)=1$ iff, for some w’ in W such that w stands in R to w’, $v_{w'}(A)=1$**

Def: A is true at w under v iff $v_w(A)=1$

[Whiteboard example 2]

# K-semantic consequence

Def: Q is a K-semantic consequence of P1,…,Pn (P1,…,Pn $|=_K$ Q) iff, for any K-model <W, R, v>, for any w in W, if P1,…, Pn are true at w under m, then Q is true at w under m

Note: The members of W typically aren't possible worlds. They can be anything! But they play the role in K-consequence that possible worlds play in logical consequence. In this sense, they may be said to "represent" worlds.

# Examples

(1) $\Box P, \Box(P \supset Q) \models_K \Box Q$

(2) $\sim\Box P \models_K \Box{\sim}P$

(3) $\sim \Diamond P \models_K \Diamond{\sim}P$

[Whiteboard example 3]

# How is K-semantic consequence related to logical consequence?

We can show that the following result is true given (Necessary Bivalence), which is what I called the “classical assumption” in seminar 2.

(Necessary Bivalence) Necessarily, every sentence meaning is either true or false.

Result 1: For any sentences P1,…,Pn, Q in M, if Q is a K-consequence of P1,…,Pn, then Q is a logical consequence of P1,…,Pn

See course website for argument.

# How is K-semantic consequence related to logical consequence? (cont)

Question: What about the converse of result 1?

If Q is a logical consequence of P1,…, Pn, is it the case that Q is a K-semantic consequence of P1,…,Pn?

# Application of Result 1

Using Result 1, we can now use facts about K-semantic consequence to establish facts about logical consequence.

Example: From (1) and Result 1, we can establish that $\Box Q$ is a logical consequence of $\Box(P \supset Q)$ and $\Box P$

# K proof-theoretic consequence

Def: Q is a K proof-theoretic consequence of P1,…Pn iff there is a **closed K-complete** tableaux whose initial list is made up of

P1,0

.
.
.

Pn, 0

~Q,0

# Definitions

Def: A tableaux is **K-complete** iff it is not possible to apply any K-rules to it

Def: A branch is **closed** iff it contains nodes of the form <A, i> and <~A,i>; Otherwise it is open.

Def: A tableaux is **closed** iff it has no open branches; otherwise it is open.

# K-rules

There are K-rules for ~, &, v, ⊃, ≡, □, and ◊

The K-rules for for ~, &, v, ⊃, ≡ are the same as that for non-modal logic (eg CSL) except we have world indices

The K-rules of □, and ◊ are listed on p. 24 Priest

[Whiteboard examples 4-6 from Priest]

# Soundness and completeness for K

Soundness Theorem: For any sentences P1,…,Pn, Q in M, if P1,…,Pn$|-_K$ Q then P1,…,Pn$|=_K$ Q

Completeness Theorem: For any sentences P1,…,Pn, Q in M if P1,…,Pn $|=_K$ Q then P1,…,Pn$|-_K$ Q

For proofs see Priest sec 2.9

# The converse of result 1

Question: Is (Converse result 1) true?

(Converse result 1) For any sentences P1,…,Pn, Q in M, if Q is a logical consequence of P1,…,Pn, then Q is a K-semantic consequence of P1,…,Pn

# The converse of result 1

Plausible answer = No!

Example: P is a logical consequence of □P, but it is not a K-semantic consequence of □P (nor is it a K proof-theoretic consequence of □P)

Solution: Come up with a stronger logic that fully captures logical consequence for the modal language M. We will attempt to do this next seminar.