Normal Modal Logic

Seminar 7

PHIL2520 Philosophy of Logic

20 November 2012

Administration

- Assignment 2: Due Monday 26 November (hand into the philosophy office by 5 PM)
- Note: **No late assignments will be accepted**
- Pick up assignments from philosophy office from wed 28 Nov
- Assignment 3: Will be handed out next week. Due Monday 3 December
- **Essay Date: Tuesday 8 January 5pm**, Email submission to danm@hku.hk (Make sure you write your name and student number in the email title and on your essay)
- Required reading for this seminar: Ch3, Priest

Schedule for rest of course

Seminar 8 (27 Nov): Normal Modal Logic, Strict Conditionals

Seminar 9 (4 Dec): Conditional Logic,

Seminar 10 (11 Dec): Many Valued Logic

18 Dec: Exam 1.30pm-3.30pm here

8 Jan: Essay Due 5pm

The language M

M is the language obtained from S* by adding '□' and '◊', where

i) '□' symbolises 'necessarily',

ii) '◊' symbolises 'possibly', and

iii) '□A' and '◊A' are sentences in M if A is a sentence in M.

K models

A K model of M is a triple <W, R,v>, where

i) W is a non-empty set (the set of “possible worlds”),

ii) R is a binary set-relation on W (the “relative possibility relation”), and

iii) v is a function mapping each pair comprising of a sentential constant p and world w to either 0 or 1 (denoted by $v_w(p)$).

v is called the interpretation wrt <W, R, v>, while R is called the **accessibility relation** wrt <W, R, v>.

K models (cont)

Given an interpretation v, v is extended to apply to all sentences in M so that it maps all sentences to either 1 or 0 in such a way that:

i) $v_w(\sim A)=1$ iff $v_w(A)=0$

ii) $v_w(A\&B)=1$ iff $v_w(A)=1$ and $v_w(B)=1$

iii) $v_w(A \vee B)=1$ iff either $v_w(A)=1$ or $v_w(B)=1$

iv) $v_w(A \supset B)=1$ iff either $v_w(A)=0$ or $v_w(B)=1$

v) $v_w(A \equiv B)=1$ iff $v_w(A)=v_w(B)$

vi) $v_w(\Box A)=1$ iff, for any w' in W such that w stands in R to w', $v_{w'}(A)=1$, and

vii) $v_w(\Diamond A)=1$ iff, for some w' in W such that w stands in R to w', $v_{w'}(A)=1$

Logical consequence and K-semantic consequence for M

Def: Q is a **logical consequence** of P1,…,Pn iff, for any meaning interpretation m, for any possible world w, if P1,…, Pn are true at w under m, then Q is true at w under m

Def: Q is a **K-semantic consequence** of P1,…,Pn (P1,…,Pn $\models_K$ Q) iff, for any K-model <W, R, v>, for any w in W, if P1,…, Pn are true at w under <W, R, v>, then Q is true at w under <W, R, v>.

Logical soundness and logical completeness

Def: Logic X is **logically sound** for language M iff: For any sentences P1,…,Pn, Q in M, if Q is a K-semantic consequence of P1,…,Pn, then Q is a logical consequence of P1,…,Pn

Def: Logic X is **logically complete** for language M iff: For any sentences P1,…,Pn, Q in M, if Q is a logical consequence of P1,…,Pn, then Q is a a K-semantic consequence of P1,…,Pn

Is K logically sound and complete for M?

Given (Necessary Bivalence), K is logically sound for M. (This was result 1.)

(Necessary Bivalence) Necessarily, every sentence meaning is either true or false.

Result 1: For any sentences P1,…,Pn, Q in M, if Q is a K-semantic consequence of P1,…,Pn, then Q is a logical consequence of P1,…,Pn

However, K is not logically complete since P is a logical consequence of □P, but it is not a K-semantic consequence of □P

A goal for modal logic

- Find a logic X which is logically sound and complete for language M
- Such a logic might be called the "true" logic for M

Note: A logic is simply a set of models or interpretations in terms of which a notion of semantic consequence can be defined.

Normal modal logics

Def: A normal modal logic is any modal logic that is obtained from K by defining semantic consequence in terms of truth preservation on K-models whose accessibility relation R satisfy certain constraints

Possible constraints

ρ (rho), reflexivity: for any w ∈ W, <w,w> ∈ R

σ (sigma), symmetry: for any w,w’ ∈ W, if <w,w’> ∈ W, then <w’,w> ∈ R

τ (tau), transitivity: for any w,w’,w’’ ∈ W, if <w,w’> ∈ W and <w’,w’’> , then <w,w’’> ∈ R

η (eta), extendability: for any w ∈ W, there is a w’ ∈ W such that <w,w’> ∈ R

The logic Kρ

Def: A ρ-model is a K-modal <W,R,v> such that R satisfies ρ

Def: Q is a **Kρ-semantic consequence** of P1,…,Pn iff, for any Kρ-model <W, R, v>, for any w in W, if P1,…, Pn are true at w under <W, R, v>, then Q is true at w under <W, R, v>.

Similar definitions can be given for Kσ, Kτ and Kη.

Combinations of constraints

The constraints ρ, σ, τ and η can be combined.

Example: A ρσ-model is a K-modal <W,R,v> such that R satisfies ρ and σ

We can therefore define semantic consequence for Kρσ, Kρτ, Kρστ etc.

Extensions

Def: Logic L1 is an extension of logic L2 iff:

For any sentences P1,…,Pn, Q in M, if Q is a L2-semantic consequence of P1,…,Pn, then Q is a L1-semantic consequence of P1,…,Pn

Result: Every normal modal logic is an extension of K

What constraints should the true logic for M satisfy?

- In order to find the true logic for M we need to determine what constraints accessibility relations should satisfy
- To answer this question we need to determine what constraints the relation of relative possibility satisfies

A popular answer

A popular answer is that the relation of relative possibility is reflexive, symmetric and transitive (that is, it satisfies ρ, σ, and τ)

Given this answer, and given (necessary bivalence), we can establish (Result 2).

It follows from (Result 2) that Kρστ is the true logic for M.

Result 2

(Result 2) For any sentences P1,…,Pn, Q in M,

Q is a logical consequence of P1,…,Pn iff Q is a a Κρστ-semantic consequence of P1,…,Pn

See course website for argument.

The specialness of logic S5

Consider the following trivial constraint:

ν (nu) triviality: for any w,w’∈W, <w,w’> ∈ W

In a ν-model, R becomes redundant: we could replace ν-models with <W,ν> pairs and drop all mention of R.

Result: Kρστ is equivalent to Kν in the sense that P1,…,Pn $|=_{K\rho\sigma\tau}$ Q iff P1,…,Pn $|=_{K\nu}$ Q (See Priest for proof.)

Because of this equivalence Kρστ and Kν are both called S5

Reason 1 to reject S5: rejecting symmetry

Combinatorialism: p is possible iff it results from recombining actual things and actually instantiated properties (Armstrong, Wittgenstein)

Ex: ‘There is a blue sheep’ is possible since the property of being a sheep can be combined with the property of being blue

Given combinatorialism, it is plausible that there can’t be more things than there actually are, but there can be less things. Given this, symmetry fails

Reason 2 to reject S5: Reject transitivity

Chisholm's Paradox:

(1) I can be transformed into a poached egg by small changes

(2) If only a small change would make me an F, then I could have been an F

(3) Relative possiblity is transitive

It follows from (1-3) that I could have been a poached egg. But this is false – I could not have been a poached egg!

Solution endorsed by David Lewis

The relative possiblity relation is not transitive.

Why isn’t relative possiblity transitive?

Lewis’s answer: possiblity is a matter of similarity (I am possibly F iff being F is not too different from how I actually am). But similarity is not transitive.

Tableaux rules for ρ, σ, τ, η

See p38-45 Priest