Solutions to Assignment 2

Q1. Show that, for any K-model $\langle W, R, v\rangle$, for any $w \in W$, and for any sentence A in $\mathcal{M}$:

$$v_w(\neg\Box A) = v_w(\Diamond\neg A)$$

Proof. Let $\langle W, R, v\rangle$ be a K-model, $w \in W$, and A be a sentence in $\mathcal{M}$. Then:

$v_w(\neg\Box A) = 1$

iff there is a $w' \in W$ such that $\underbrace{\langle w, w'\rangle \in R \text{ and } v_{w'}(A) = 0}$

iff there is a $w' \in W$ such that $\underbrace{\langle w, w'\rangle \in R \text{ and } v_{w'}(\neg A) = 1}$

iff $v_w(\Diamond\neg A) = 1$

Q2.

~~2~~ j) ~~Sh~~ Establish

$\Box(A \supset B) \vdash \Diamond A \supset \Diamond B$

Proof.

$\Box(A \supset B), 0$

$\neg(\Diamond A \supset \Diamond B), 0$ ✓

$\Diamond A, 0$ ✓

$\neg \Diamond B, 0$ ✓

$\Box \neg B, 0$

$0 r 1$

$A, 1$

$A \supset B, 1$ ✓

$\neg B, 1$

$\neg A, 1$ $\quad$ $B, 1$

× $\quad$ ×

Q3

2) Establish $\vdash \Box A \equiv \Box(\neg A \supset A)$

Proof. $\checkmark\ \neg[\Box A \equiv \Box(\neg A \supset A)],\ 0$

$\Box A, 0$

$\checkmark\ \neg\Box(\neg A \supset A), 0$

$\checkmark\ \Diamond\neg(\neg A \supset A), 0$

$0r1$

$\checkmark\ \neg(\neg A \supset A), 1$

$\neg A, 1$

$\neg A, 1$

$A, 1$

$\times$

$\neg\Box A, 0\ \checkmark$

$\Box(\neg A \supset A), 0$

$\Diamond\neg A, 0\ \checkmark$

$0r2$

$\neg A, 2$

$\neg A \supset A, 2\ \checkmark$

$\checkmark\ \neg\neg A, 2$ | $A, 2$

$A, 2$ | $\times$

$\times$

Q4 Establish and find and verify a countermodel.

$\nvdash \Box(p \vee q) \supset (\Box p \vee \Box q)$

Proof. $\neg[\Box(p \vee q) \supset (\Box p \vee \Box q)], 0$ ✓

$\Box(p \vee q), 0$

$\neg(\Box p \vee \Box q), 0$ ✓

$\neg\Box p, 0$ ✓

$\neg\Box q, 0$ ✓

$\Diamond\neg p, 0$ ✓

$\Diamond\neg q, 0$ ✓

$0r1$

$\neg p, 1$

$0r2$

$\neg q, 2$

$p \vee q, 1$

$p \vee q, 2$

$p, 1$ ×	$q, 1$

$p, 2$ ○	$q, 2$ ×

Countermodel:

Let $W = \{ w_0, w_1, w_2 \}$

$R = \{ \langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle \}$

$V_{w_2}(p) = 1 \quad V_{w_2}(q) = 0, \quad V_{w_1}(p) = 0,$ ~~w~~

$V_{w_1}(q) = 1$

Verify this is a countermodel

We have

$$\begin{array}{l} \neg\Box q \qquad \neg(\Box p \vee \Box q) \\ \neg\Box p \\ \Box(p \vee q) \end{array}$$

$$\begin{tikzcd} & w_0 \arrow[dl] \arrow[dr] & \\ \begin{array}{l} w_1 \\ \neg p \\ q \end{array} & & \begin{array}{l} w_2 \\ p \\ \neg q \end{array} \end{tikzcd}$$

Hence '$\Box(p \vee q) \supset (\Box p \vee \Box q)$' is false

5. Establish and find and verify a countermodel

~~$\nvdash (\Diamond p \supset \Diamond\Diamond p)$~~ $\nvdash \Box p \supset \Box\Box p$

Proof.

$\neg[\Box p \supset \Box\Box p], 0$ ✓

$\Box p, 0$

$\neg\Box\Box p, 0$ ✓

$\Diamond\neg\Box p, 0$ ✓

$0r1$

$\neg\Box p, 1$ ✓

$\Diamond\neg p, 1$

$1r2$

$\neg p, 2$

$p, 1$

$\bigcirc$

Hence $\nvdash (\Box p \supset \Box\Box p)$

Countermodel

Let $W = \{ w_1, w_2, w_3 \}$

$R = \{ \langle w_0, w_1 \rangle, \langle w_1, w_2 \rangle \}$

$V_{w_1}(p) = 1$, $V_{w_2}(p) = 0$

$$\begin{array}{ccccc} w_0 & \longrightarrow & w_1 & \longrightarrow & w_2 \\ \Box p & & p & & \neg p \\ \neg \Box\Box p & & & & \end{array}$$

Hence $(\Box p \supset \Box\Box p)$ is ~~t~~ false at w_0 under
$\langle W, R, v \}$