

Topic 1: Answers to Exercises

Exercise 1.1.4a

Show the following using only &E, \rightarrow E, or A.

$(A \& B) \vdash A$

1	1. $(A \& B)$	A
1	2. A	1 &E

$((A \vee B) \& (B \rightarrow A)) \vdash A$

Cannot be shown with only these three rules.

$(A \rightarrow (B \& C)), A \vdash B$

1	1. $(A \rightarrow (B \& C))$	A
2	2. A	A
1, 2	3. $(B \& C)$	1, 2 \rightarrow E
1, 2	4. B	3 &E

$A, B \vdash (A \& B)$

Cannot be shown with only these three rules.

$A, (A \rightarrow (B \rightarrow (C \rightarrow (D \rightarrow (E \rightarrow F)))) \vdash F$

Cannot be shown.

Exercise 1.2.1a

Yes, this is a correct derivation.

Exercise 1.2.1b

Show $(A \& B) \vdash (B \& A)$

1	1. $(A \& B)$	A
1	2. B	1 &E

1	3. A	1 &E
1	4. (B&A)	2,3 &I

Exercise 1.2.1c

There is no longest derivation in our system.
Any derivation can be lengthened, for example
by using Rule A.

Exercise 1.2.2a

Show $(A \leftrightarrow B), (B \& C) \vdash A$

1	1. $(A \leftrightarrow B)$	A
2	2. $(B \& C)$	A
2	3. B	2 &E
1	4. $((A \rightarrow B) \& (B \rightarrow A))$	1 \leftrightarrow E
1	5. $(B \rightarrow A)$	4 &E
1, 2	6. A	3, 5 \rightarrow E

Show $(A \rightarrow A) \vdash (A \leftrightarrow A)$

1	1. $(A \rightarrow A)$	A
1	2. $((A \rightarrow A) \& (A \rightarrow A))$	1, 1 &I
1	3. $(A \leftrightarrow A)$	2 \leftrightarrow I

Show $(A \leftrightarrow B), (B \leftrightarrow C), A \vdash C$

1	1. $(A \leftrightarrow B)$	A
2	2. $(B \leftrightarrow C)$	A
3	3. A	A
1	4. $((A \rightarrow B) \& (B \rightarrow A))$	1 \leftrightarrow E
1	5. $(A \rightarrow B)$	4 &E
1, 3	6. B	3, 5 \rightarrow E
2	7. $((B \rightarrow C) \& (C \rightarrow B))$	2 \leftrightarrow E
2	8. $(B \rightarrow C)$	7 &E
1, 2, 3	9. C	6, 8 \rightarrow E

Exercise 1.2.3a

There is no example.

Any formula which is assumed (written down
in a derivation using Rule A) is derived (written

down in a derivation).

Exercise 1.2.3b

Yes, this is a correct derivation.

Exercise 1.2.3c

This is not a correct derivation.

On line 5, Rule $\rightarrow I$ is misused because line 3 is not an assumed.

Exercise 1.2.3d

Show $(P \rightarrow (Q \rightarrow R)) \vdash ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

1	1. $(P \rightarrow (Q \rightarrow R))$	A
2	2. $(P \rightarrow Q)$	A
3	3. P	A
2, 3	4. Q	2, 3 $\rightarrow E$
1, 3	5. $(Q \rightarrow R)$	1, 3 $\rightarrow E$
1, 2, 3	6. R	4, 5 $\rightarrow E$
1, 2	7. $(P \rightarrow R)$	3, 6 $\rightarrow I$
1	8. $((P \rightarrow Q) \rightarrow (P \rightarrow R))$	2, 7 $\rightarrow I$

Show $\vdash (A \rightarrow A)$

1	1. A	A
	2. $(A \rightarrow A)$	1, 1 $\rightarrow I$

Show $\vdash ((A \& B) \rightarrow A)$

1	1. $(A \& B)$	A
1	2. A	1 &E
	3. $((A \& B) \rightarrow A)$	1, 2 $\rightarrow I$

Exercise 1.3.1a

Show $(B \& \sim B) \vdash A$

1	1. $(B \& \sim B)$	A
1	2. B	1 &E
1	3. $\sim B$	1 &E
1	4. $(B \vee A)$	2 $\vee I$
1	5. A	3, 4 $\vee E$

Exercise 1.3.1b

Yes, for example:

1	1. $(A \vee A)$	A
2	2. $(A \rightarrow B)$	A
1,2	3. $(B \vee B)$	1,2,2 PC

Exercise 1.3.1c

Show $((P \& P) \vee (Q \& Q)) \vdash (P \vee Q)$

1	1. $((P \& P) \vee (Q \& Q))$	A
2	2. $(P \& P)$	A
2	3. P	2 &E
	4. $((P \& P) \rightarrow P)$	2,3 \rightarrow I
5	5. $(Q \& Q)$	A
5	6. Q	5 &E
	7. $((Q \& Q) \rightarrow Q)$	5,6 \rightarrow I
1	8. $(P \vee Q)$	1,4,7 PC

Exercise 1.3.1d

Show $(P \vee P), (P \rightarrow Q) \vdash (P \vee Q)$

1	1. $(P \vee P)$	A
2	2. $(P \rightarrow Q)$	A
3	3. P	A
	4. $(P \rightarrow P)$	3, 3 \rightarrow I
1, 2	5. $(P \vee Q)$	1, 2, 4 PC

Exercise 1.3.2a

Show $(\sim A \vee \sim B) \vdash \sim(A \& B)$

1	1. $(\sim A \vee \sim B)$	A
2	2. $(A \& B)$	A
2	3. A	2 &E
4	4. $\sim A$	A
2, 4	5. $(A \& \sim A)$	3, 4 &I
2	6. $\sim \sim A$	4, 5 \sim I
1, 2	7. $\sim B$	1, 6 \vee E
2	8. B	2 &E
1, 2	9. $(B \& \sim B)$	7, 8 &I
1	10. $\sim(A \& B)$	2, 9 \sim I

Exercise 1.3.2b

Show $\sim \sim A \vdash A$

1	1. $\sim \sim A$	A
2	2. $\sim A$	A

1, 2	3. ($\sim A \& \sim \sim A$)	1, 2 &I
1	4. A	2, 3 $\sim E$

Show $\vdash (\sim \sim A \rightarrow A)$

1	1. $\sim \sim A$	A
2	2. $\sim A$	A
1, 2	3. ($\sim A \& \sim \sim A$)	1, 2 &I
1	4. A	2, 3 $\sim E$
	5. ($\sim \sim A \rightarrow A$)	1, 4 $\rightarrow I$

Show $\sim(P \vee Q) \vdash (\sim P \& \sim Q)$

1	1. $\sim(P \vee Q)$	A
2	2. P	A
2	3. (P \vee Q)	2 $\vee I$
1, 2	4. ((P \vee Q) $\&$ $\sim(P \vee Q)$)	1, 3 &I
1	5. $\sim P$	2, 4 $\sim I$
6	6. Q	A
6	7. (P \vee Q)	6 $\vee I$
1, 6	8. ((P \vee Q) $\&$ $\sim(P \vee Q)$)	1, 7 &I
1	9. $\sim Q$	6, 8 $\sim I$
1	10. ($\sim P \& \sim Q$)	5, 9 &I

Exercise 1.4.2

Yes. Other possible rules are: $\rightarrow E$, $\&E$ and $\vee E$.

Exercise 1.4.4

$\sim((A \& B) \vee C) \vdash (\sim(A \& B) \& \sim C)$

Just take the derivation for $\sim(P \vee Q) \vdash (\sim P \& \sim Q)$, and substitute every occurrence of "P" with "(A & B)" and "Q" with "C".

1	1. $\sim((A \& B) \vee C)$	A
2	2. (A & B)	A
2	3. ((A & B) \vee C)	2 $\vee I$
1, 2	4. (((A & B) \vee C) $\&$ $\sim((A \& B) \vee C)$)	1, 3 &I
1	5. $\sim(A \& B)$	2, 4 $\sim I$
6	6. C	A
6	7. ((A & B) \vee C)	6 $\vee I$
1, 6	8. (((A & B) \vee C) $\&$ $\sim((A \& B) \vee C)$)	1, 7 &I
1	9. $\sim C$	6, 8 $\sim I$

1 10. $(\sim(A \& B) \& \sim C)$

5, 9 & I