Exercise 3.2.1b
Explain why "Fa" does not entail "∀xFx".

Consider an interpretation under which "Fa" is true but "Fb" is false. Under this interpretation, "∀xFx" is false (because there exists an element in the domain to which the predicate F does not apply). Since there is an interpretation under which "Fa" is true and "∀xFx" is false, "Fa" does not entail "∀xFx".

Exercise 3.2.1c
Explain why line 3 in the last example violates Rule ∀I.

Rule ∀I says that for any variable v and constant c, if you have derived φv/c, and c does not occur in φ, and c does not occur in anything φv/c depends on, and ∀vφ is a well-formed formula of MPL, then you can write down ∀vφ, depending on everything φv/c depends on. In the example, φ is "(Fa→Fx)", v is "x", c is "a", and φv/c is "(Fa→Fa)". The formula on line 3 is "∀x(Fa→Fx)" which violates the restriction that c does not occur in φ, because clearly, "a" *does* occur in "(Fa→Fx)".

Exercise 3.2.1d
Show that it is not the case that |= ∀x(Fa→Fx).

If |= ∀x(Fa→Fx), then "∀x(Fa→Fx)" is true under all interpretations. Consider an interpretation in which "Fa" is true but "Fb" is false. Under this interpretation, "(Fa→Fb)" is false. So "∀x(Fa→Fx)" is false under this interpretation as well. So "∀x(Fa→Fx)" is not true under all interpretations, and hence it is not the case that |= ∀x(Fa→Fx).

Exercise 3.2.2a
What restriction of the rule is violated on line 3 of the last example?

The rule says that for any variable v and constant c, if you have derived ∃vφ, assumed φv/c, and derived ψ, and c does not occur in ψ, φ, or anything ψ depends on (except φv/c), then you can write down ψ a second time, depending on everything ∃vφ and the first ψ depend on, except the assumption φv/c. In this example, φ is "Fx", v is "x", c is "a", φv/c is "Fa", and ψ is "Fa". Line 3 violates the restriction that c does not occur in ψ – because "a" *does* occur in "Fa".

Exercise 3.2.3a
Show that "∀x(Fx→Fx)" is valid in two different ways.

"∀x(Fx→Fx)" is true under every interpretation. So "∀x(Fx→Fx)" is valid.

"∀x(Fx→Fx)" is derivable using MPL natural deduction:

1	1. Fa	A
	2. (Fa→Fa)	1, →I
	3. ∀x(Fx→Fx)	2, ∀I

Since "∀x(Fx→Fx)" is derivable with no dependencies, and the system is sound, "∀x(Fx→Fx)" is valid.

Exercise 3.2.3b
Suppose you try to find a derivation of a certain MPL formula, but you do not succeed in finding a derivation. Does it follow that that formula is not valid?

No. If there is no derivation, the formula is not valid. But not succeeding in finding a derivation does not mean that there isn't a derivation.

Exercise 3.2.4a

∀xHx ⊢ Ha

1	1) ∀xHx	A
1	2) Ha	1 ∀E

∀xHx, (Hc → ∃xGx) ⊢ ∃xGx

1	1) ∀xHx	A
2	2) (Hc→∃xGx)	A
1	3) Hc	1 ∀E
1, 2	4) ∃xGx	2, 3 →E

∀x(Hx→Mx), ∀xHx ⊢ ∀xMx

1	1) ∀x(Hx→Mx)	A
2	2) ∀xHx	A
1	3) (Ha→Ma)	1 ∀E
2	4) Ha	2 ∀E
1, 2	5) Ma	3, 4 →E
1, 2	6) ∀xMx	5 ∀I

Ha ⊢ ∃yHy

1	1) Ha	A
2	2) ∃yHy	1 ∃I

(Ab → Dc) ⊢ (Ab → ∃xDx)

1	1) (Ab→Dc)	A
2	2) Ab	A

1, 2	3) Dc	1, 2 →E
1, 2	4) ∃xDx	3 ∃I
1	5) (Ab→∃xDx)	2, 4 →I

∃x(Fx & Gx) ⊢ ∃xFx

1	1) ∃x(Fx&Gx)	A
2	2) (Fa&Ga)	A
2	3) Fa	2 &E
2	4) ∃xFx	3 ∃I
1	5) ∃xFx	1, 2, 4 ∃E

∀x(Fx→∀yGy) ⊢ ∀x∀y(Fx→Gy)

1	1) ∀x(Fx→∀yGy)	A
1	2) (Fa→∀yGy)	1 ∀E
3	3) Fa	A
1, 3	4) ∀yGy	2, 3 →E
1, 3	5) Gb	4 ∀E
1	6) (Fa→Gb)	3, 5 →I
1	7) ∀y(Fa→Gy)	6 ∀I
1	8) ∀x∀y(Fx→Gy)	7 ∀I

∀x(Px→Qx), ∀x(Qx→Px) ⊢ ∀x(Px↔Qx)

1	1) ∀x(Px→Qx)	A
2	2) ∀x(Qx→Px)	A
1	3) (Pa→Qa)	1 ∀E
2	4) (Qa→Pa)	2 ∀E
1, 2	5) ((Pa→Qa)&(Qa→Pa))	3, 4 &I
1, 2	6) (Pa↔Qa)	5 ↔I
1, 2	7) ∀x(Px↔Qx)	6 ∀I

∃x~Px ⊢ ~∀xPx

1	1) ∃x~Px	A
2	2) ∀xPx	A
3	3) ~Pa	A
2	4) Pa	2 ∀E
5	5) ~(Qb&~Qb)	A
2, 3	6) (Pa&~Pa)	3, 4 &I
2, 3	7) (Qb&~Qb)	5, 6 ~E
1, 2	8) (Qb&~Qb)	1, 3, 7 ∃E
1	9) ~∀xPx	2, 8 ~I

(∃xPx→∀x(Qx→Rx)), (Pa&Qa) ⊢ Ra

1	1) (∃xPx→∀x(Qx→Rx))	A
2	2) (Pa&Qa)	A
2	3) Pa	2 &E
2	4) ∃xPx	3 ∃I
1, 2	5) ∀x(Qx→Rx)	1, 4 →E
2	6) Qa	2 &E
1, 2	7) (Qa→Ra)	5 ∀E
1, 2	8) Ra	6, 7 →E

(∀x(Px→Qx)→∃x(Rx&Sx)), (∀x(Px→Sx)&∀x(Sx→Qx)) ⊢ ∃xSx

1	1) (∀x(Px→Qx)→∃x(Rx&Sx))	A
2	2) (∀x(Px→Sx)&∀x(Sx→Qx))	A
2	3) ∀x(Px→Sx)	2 &E
2	4) ∀x(Sx→Qx)	2 &E
5	5) Pa	A
2	6) (Pa→Sa)	3 ∀E
2, 5	7) Sa	5, 6 →E
2	8) (Sa→Qa)	4 ∀E
2, 5	9) Qa	7, 8 →E
2	10) (Pa→Qa)	5, 9 →I
2	11) ∀x(Px→Qx)	10 ∀I
1, 2	12) ∃x(Rx&Sx)	1, 11 →E
13	13) (Rb&Sb)	A
13	14) Sb	13 &E
13	15) ∃xSx	14 ∃I
1, 2	16) ∃xSx	12, 13, 15 ∃E

(∀x(Px&~Qx)→∃xRx), ~∃x(Qx∨Rx) ⊢ ~∀xPx

1	1) (∀x(Px&~Qx)→∃xRx)	A
2	2) ~∃x(QxvRx)	A
3	3) ∀xPx	A
4	4) Qa	A
4	5) (QavRa)	4 vI
4	6) ∃x(QxvRx)	5 ∃I
2, 4	7)(∃x(QxvRx)&~∃x(QxvRx))	2, 6 &I
2	8) ~Qa	4, 7 ~I
3	9) Pa	3 ∀E
2, 3	10) (Pa&~Qa)	8, 9 &I

2, 3	11) ∀x(Px&~Qx)	10 ∀I
1, 2, 3	12) ∃xRx	1, 11 →E
13	13) Rb	A
13	14) (QbvRb)	13 vI
13	15) ∃x(QxvRx)	14 ∃I
1, 2, 3	16) ∃x(QxvRx)	12, 13, 15 ∃E
1, 2, 3	17) (∃x(QxvRx)&~∃x(QxvRx))	2, 16 &I
1, 2	18) ~∀xPx	3, 17 ~I

(∃x~Px→∀x~Qx), (∃x~Px→∃xQx), ∀x(Px→Rx) ⊢ ∀xRx

1	1) (∃x~Px→∀x~Qx)	A
2	2) (∃x~Px→∃xQx)	A
3	3) ∀x(Px→Rx)	A
4	4) ~Pa	A
4	5) ∃x~Px	4 ∃I
1, 4	6) ∀x~Qx	1, 5 →E
2, 4	7) ∃xQx	2, 5 →E
8	8) Qa	A
9	9) ~(Sb&~Sb)	A
1, 4	10) ~Qa	6 ∀E
1, 4, 8	11) (Qa&~Qa)	8, 10 &I
1, 4, 8	12) (Sb&~Sb)	9, 11 ~E
1, 2, 4	13) (Sb&~Sb)	7, 8, 12 ∃E
1, 2	14) Pa	4, 13 ~E
3	15) (Pa→Ra)	3 ∀E
1, 2, 3	16) Ra	14, 15 →E
1, 2, 3	17) ∀xRx	16 ∀E

~∃x(Px∨Qx), (∃xRx→∃xPx), (∃xSx→∃xQx) ⊢ ~∃x(Rx∨Sx)

1	1) ~∃x(PxvQx)	A
2	2) (∃xRx→∃xPx)	A
3	3) (∃xSx→∃xQx)	A
4	4) ∃x(RxvSx)	A
5	5) Ra	A
5	6) ∃xRx	5 ∃I
2, 5	7) ∃xPx	2, 6 →E
8	8) Pb	A
8	9) (PbvQb)	8 vI
8	10) ∃x(PxvQx)	9 ∃I
2, 5	11) ∃x(PxvQx)	7,8,10∃E
1, 2, 5	12) (∃x(PxvQx)&~∃x(PxvQx))	1, 11 &I
1, 2	13) ~Ra	5, 12 ~I
14	14) Sa	A
14	15) ∃xSx	14 ∃I
3, 14	16) ∃xQx	3, 15 →E
17	17) Qa	A
17	18) (PavQa)	17 vI
17	19) ∃x(PxvQx)	18 ∃I
3, 14	20) ∃x(PxvQx)	16, 17, 19 ∃E
1, 3, 14	21) (∃x(PxvQx)&~∃x(PxvQx))	1, 20 &I
1, 3	22) ~Sa	14, 21 ~I
23	23) (RavSa)	A
24	24) ~(Tb&~Tb)	A
1, 2, 23	25) Sa	13, 23 vE
1 ,2, 3, 23	26) (Sa&~Sa)	22, 25 &I
1, 2, 3, 23	27) (Tb&~Tb)	24, 26 ~E
1, 2, 3, 4	28) (Tb&~Tb)	4, 23, 27 ∃E
1, 2, 3	29) ~∃x(RxvSx)	4, 28 ~I